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On the solvability of the bouncer model

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Abstract. Two new exact solutions for the bouncer model are found and their relations to some solvable and non-solvable Lie algebras are shown.

1. Introduction

We consider here a model, named after Gibbs [1] the bouncer, i.e. a particle falling down in a constant gravitational field on a plate moving according to a given function of time $L(t)$. It was suggested by Zaslavsky [2] as an alternative to the well known Fermi–Ulam model [3] of cosmic ray acceleration.

It is worth noting that the bouncer model has some experimental realizations. The first apparatus, consisting of an ordinary loudspeaker, a function generator and a small ball, has been constructed by Pierański [4]. His experimental studies were continued with a modified apparatus [5, 6] and the results compared with calculations of classical maps which can be exactly iterated for any function $L(t)$. For a periodic function $L(t)$ this has been done for both one and two frequencies [5, 7]. Usually, the ball-plate collisions are assumed to be perfectly elastic, though a completely inelastic case has also been studied [8]. For completeness we should also mention a two-ball variant of the model [9]. All the studies have shown that the bouncer is an example of a chaotic system when the plate is assumed to oscillate periodically with some frequency.

Quantization of the bouncer model has been performed quite recently [10]. In order to do that one needs to solve the time-dependent Schrödinger equation with the Hamiltonian ($\hbar = 1$)

$$H(x) = -\frac{1}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + mgx \tag{1}$$

(where m is the mass of the ball and g the acceleration due to gravity) subject to the boundary and normalization conditions:

$$\forall t : \left[\Psi(x,t) = 0 \quad \text{for} \quad x = L(t) \qquad \text{and} \quad \int_{L(t)}^{\infty} |\Psi(x,t)|^2 \, \mathrm{d}x = 1 \right]. \tag{2}$$

The only known exact solution of the model obeying (2) has been found [11] so far for $L(t) \equiv L_1(t) = at^2 + bt + c$ (a, b, c = constant). In such a case the wavefunctions are given by the Airy functions $\mathrm{Ai}(\cdots)$. The periodic motion of the plate, necessary for the classically chaotic motion to exist, has been represented in the quantum version of the model [10] by concatenating different parabolas $L_1(t)$. Matching suitably the parabolas one can construct a periodic motion of the plate from a number of convex–convex–$\cdots$, concave–concave–$\cdots$ or convex–concave–$\cdots$ curves thus leading to three variants of the model called the ‘convex’, the ‘concave’ or the ‘smooth’ bouncers [10], respectively.

The necessity of a such procedure follows from the fact that finding exact analytical solutions for wavefunctions is not possible for classical mechanical systems that are chaotic. Moreover, direct numerical integration of the time-dependent Schrödinger equation (TDSE) with oscillating perturbations cannot be performed with accuracy required for a trustworthy analysis of results. Thus, any new exact solution of the TDSE for the discussed model is a valuable one since it allows to have at our disposal an additional fully tractable physical model for studying the classical–quantum correspondence. This was the first motivation for the present study.

The second reason for dealing with the model was the recent controversy concerning the existence of the exact solutions. We have proved [12] that the solutions proposed and used in [13] are not correct. The question whether or not any new solutions are can be derived at all is partly answered here in sections 2 and 3.

2. Solutions of the model

In what follows we shall derive two new exact solutions of the quantum version of the bouncer model described by equations (1) and (2). To this end, let us start with the most general linear transformation

$$x = s(t)y + L(t) \tag{3}$$

which replaces a moving boundary in the x-space with the fixed one in the y-space. Possible choices for a real function $s(t)$ will be found below and here we require $s(t) \neq 0$ for any time. Now, using equations (1) and (3), from the time-dependent Schrödinger equation we get

$$\mathrm{i}s^2(t)\frac{\partial \Psi}{\partial t} = -\frac{1}{2m}\frac{\partial^2\Psi}{\partial y^2} + \mathrm{i}s\dot{s}y\frac{\partial \Psi}{\partial y} + \mathrm{i}s\dot{L}\frac{\partial \Psi}{\partial y} + mg(ys^3 + Ls^2)\,\Psi(y,\ t) \tag{4}$$

where the dot over the symbols denotes the time derivative and, instead of (2) we now have

$$\forall t : \left[\Psi(y,t) = 0 \quad \text{for} \quad y = 0 \qquad \text{and} \quad \int_0^\infty |\Psi(y,t)|^2\,\mathrm{d}y = 1\right]. \tag{5}$$

If the solution $\Psi(y,t)$ is proposed in the form

$$\Psi(y,t) = (2/s)^{\frac{1}{2}}\exp\left[\mathrm{i}m\left(s\dot{L}y + \frac{1}{2}\int_0^t \dot{L}^2\,\mathrm{d}t - g\int_0^t L\,\mathrm{d}t\right) + \frac{\mathrm{i}m}{2}s\dot{s}y^2\right]\Phi(y,t) \tag{6}$$

then equation (4) can be reduced to the simpler equation

$$\mathrm{i}s^2(t)\frac{\partial \Phi}{\partial t} = -\frac{1}{2m}\frac{\partial^2\Phi}{\partial y^2} + \left[\frac{1}{2}ms^3\ddot{s}y^2 + m(\ddot{L}+g)s^3y\right]\Phi(y,\ t)\,. \tag{7}$$

It is obvious at this stage that the solvability of the model is guaranteed when

$$s^3\ddot{s} = C_1 = \text{constant} \tag{8a}$$

$$(\ddot{L}+g)s^3 = C_2 = \text{constant}\,. \tag{8b}$$

Equation (8a) has the exact solution

$$s(t) \;\equiv\; s_2(t) = \sqrt{at^2 - 2bt + c} \tag{9}$$

with $ac - b^2 = C_1$ and we have $s_2(t) \neq 0$ for $C_1 > 0$. Using equation (9) in (8*b*) and then integrating the latter twice, we get

$$L(t) \equiv L_2(t) = \frac{C_2}{C_1}\sqrt{at^2 - 2bt + c} - \frac{1}{2}gt^2 + \alpha t + \beta\,. \tag{10}$$

The case of $C_1 = 0$ has to be considered separately. Now from (8*a*) we have

$$s(t) \ \equiv \ s_3(t) = A + Bt \qquad s_3(t) \neq 0 \tag{11}$$

and from (8*b*) finally

$$L(t) \equiv L_3(t) = \frac{C_2}{2B^2(A + Bt)} - \frac{1}{2}gt^2 + Dt + E\,. \tag{12}$$

In equations (10)–(12) $\alpha,\ \beta,\ A,\ B,\ D$ and E are integration constants.

For the case of $C_1 = 0$ the solutions of (7) are given by the Airy functions $\mathrm{Ai}(\cdots)$ and in the case of $C_1 > 0$ the transformations

$$\Phi(y,\ t) = \exp\left[-\mathrm{i}E_n \int_0^t \frac{\mathrm{d}t'}{s_2^2(t')}\right]\varphi(y) \tag{13}$$

and

$$\xi = \sqrt{2m\sqrt{C_1}}\,\left(y + \frac{C_2}{C_1}\right) \tag{14}$$

reduce equation (7) to an equation for the parabolic cylinder functions in its standard form [14]. In both cases the final explicit wavefunctions obeying conditions (2) or (5) can be written without any effort and that is why we omit them here.

We have thus proved that the bouncer model is a solvable one if the plate is moving according to the functions $L_2(t)$ and $L_3(t)$, in addition to the $L_1(t)$ found earlier.

3. Lie algebraic considerations

We would like to point out now that the problem of solvability of our model can be posed in terms of some Lie algebras and the related Wei–Norman [15] or Magnus [16] methods.

Introducing the scaled time τ as

$$\tau(t) = \int_0^t \frac{\mathrm{d}t'}{s^2(t')} \tag{15}$$

we get equations (4) and (7) in the form of standard time-dependent Schrödinger equations with the Hamiltonians

$$\begin{aligned} H_s(y,\ \tau) &= -\frac{1}{2m}\partial_{yy} \ + a_2(\tau)y\partial_y + a_3(\tau)\partial_y + a_4(\tau)y + a_5(\tau) \\ &= \sum_{i=1}^{5} a_i(\tau)H_i \end{aligned} \tag{16}$$

and

$$H_n(y,\ \tau) = -\frac{1}{2m}\partial_{yy} + b_2(\tau)y^2 + b_3(\tau)y\,. \tag{17}$$

The functions $a_i(\tau)$ and $b_i(\tau)$ can easily be found with the help of (4), (7) and (15) when a simple relation $t = t(\tau)$ exists, as for $s_k(t)$ $(k = 1,\ 2,\ 3)$ given above, however, detailed forms of a_i and b_i are not important here.

Observe now that $H_s(y,\tau)$ generates a solvable Lie algebra $K_s = \{\partial_{yy}, y\partial_y, \partial_y, y, 1\}$ since the elements of K_s, which are the result of commutation of two Lie elements, form derived algebras with $K_s^{(h)} = \{0\}$ for $h = 3$. If $h = 1$ we have $K_0 = \{\partial_{yy}, \partial_y, y, 1\}$ and this is a nilpotent Lie algebra which is obviously a subalgebra of K_s. When K_s is supplemented by the term $\sim y^2$, i.e. $K_n = \{\partial_{yy}, y\partial_y, \partial_y, y^2, y, 1\}$, we get a non-solvable Lie algebra which corresponds to the Hamiltonian $H_n(y,\tau)$. This time for the derived algebras we have $K_n = K_n^{(1)} = K_n^{(2)} = \cdots = K_n^{(h)}$ for any integer h.

Thus, two solvable cases of the bouncer model correspond to the nilpotent Lie algebras K_0, one in the space (y,t) of the Schödinger equation with $s(t) = s_1(t) = 1$ and $L(t) = L_1(t)$, and the other one in the space (y,τ) with $C_1 = 0$, $C_2 \neq 0$, i.e. with $s(t) = s_3(t)$ and $L(t) = L_3(t)$. The third *solvable* case corresponds to the *non-solvable* algebra K_n in the space (y,τ) for $C_1 > 0$, $C_2 \neq 0$ with $s(t) = s_2(t)$ and $L(t) = L_2(t)$.

The algebra K_n is called non-solvable since within the scope of the Lie algebraic methods like, e.g., the Wei–Norman method [15], it leads to some system of nonlinear equations which, in general, cannot be solved by quadrature. The simple approach developed above avoids this difficulty.

Formal solutions to the Schrödinger equation can be expressed as $\Psi(y,\tau) = U(\tau,0)\Psi(y,0)$, where the exact form of the evolution operator $U(\tau,0)$ depends on the method applied. In the case of the well known formalism developed by Wei and Norman [15] we can write, e.g., for $H_s(y,\tau)$ of (16) the evolution operator as $U(\tau,0) = \prod_{i=1}^{5}\exp[g_i(\tau)H_i]$ where $g_i(\tau)$ are solutions of some nonlinear differential equations. The reader can easily check that $\Psi(y,\tau)$ derived in this way obeys the Schrödinger equation for any τ-dependence of the functions $a_i(\tau)$ in (16). Unfortunately, only one member of the family of solutions obeys the boundary conditions (5). This is the case for $L(t) = L_1(t)$, i.e. $s(t) = s_1(t) = 1$. A very similar situation appears for $C_1 = 0$ in (8*a*), i.e. for $b_2(\tau) = 0$ in (17). Again, the formally exact wavefunctions can be given for any function $b_3(\tau)$ in (17), however, there is only one solution obeying (5). This is the case for $L(t) = L_3(t)$. For the full Hamiltonian (17) the related algebra is non-solvable and finding any solution of the time-dependent Schrödinger equation obeying the boundary conditions (5) is not possible within the Wei–Norman method. The simple approach developed in this work leads without such difficulties to another exact physical solution for $L(t) = L_2(t)$.

The conclusions hold true also for other forms of the evolution operator. One can use its Magnus [16] or, e.g., Fer [17] representations but within the transformation (3) and the Lie algebras discussed here no further exact physical solutions of the bouncer model can be found. The main troublemaker in this task is the operator ∂_y closely related to the momentum operator, a generator of the translation operation. In the restricted Hilbert space, as for our model, the momentum operator is not, in general, a self-adjoint one.

Failing to pay sufficient attention to the latter point led in the past to erroneous results as shown in [12]. A deeper insight into the reasons why some wavefunctions cannot fulfil boundary conditions on a semi-axis has been presented in an elegant way in the very recent paper [18].

4. Conclusions

In summary, we have proved the solvability of the bouncer model in two more cases. For each of them the 'convex', 'concave' and 'smooth' variants can be considered as discussed in the introduction. A number of free parameters in the functions $L_2(t)$ and $L_3(t)$ allows one to conveniently model the motions of the bouncer's plate within a fixed interval of time. Of special value is the case of $L_3(t)$. Computer programmes and techniques used for the 'parabolic' (i.e. with $L(t) = L_1(t)$) bouncer model [10] can also be used with simple modifications for the $L_3(t)$ case since the solutions of the TDSE are represented in both cases by the same Airy functions. We plan to do the calculations in future and hope they will enrich our knowledge on the properties of quantum systems which are classically chaotic. The case of $L_2(t)$ is much more difficult since the solutions of the TDSE are now given in terms of the parabolic cylinder functions which are not so easy to handle numerically as the Airy functions.

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